On the Primitive Groups of Class Ten.

By W. A. Manning.

Let $s_1, s_2, \ldots.$ be a complete set of conjugate substitutions of prime order (p) in a primitive group G. They generate a transitive group. A certain number (λ) of these conjugates may generate an intransitive subgroup I_1. We wish to prove, before taking up the groups of class 10, that a *substitution can always be found in the series $s_1, s_2, \ldots.$ which connects the letters of any given set of intransitivity of I_1 with the letters of some other set.*

Suppose that the theorem is not true. Take a particular transitive set of $I_1 \equiv \{s_1, s_2, \ldots. s_\lambda\}$ in the letters $a_1, a_2, \ldots.$ Choose from the series $s_1, \ldots.$ a substitution $s_{\lambda+1}$ which connects $a_1, \ldots.$ with new letters, and which, of all the substitutions connecting the letters $a_1, \ldots.$ with new letters α, has the fewest new letters in the cycles with $a_1, \ldots.$ There may be a number of such substitutions. Select that one which connects the fewest new letters β with the letters b. Now consider $I_2 = \{I_1, s_{\lambda+1}\}$. If no substitution of the series $s_1, \ldots.$ connects the extended set $a_1, \ldots.$ with another transitive set of I_2, we take $s_{\lambda+2}$ as we did $s_{\lambda+1}$, and continue in this way until we have an intransitive subgroup I_{e-1} and a substitution S_e which connects the extended set $a_1, \ldots.$ with some other set of I_{e-1}. It is now essential to consider closely the substitution s_{e-1} by which we pass from I_{e-2} to I_{e-1}. Let the letters of the first set be $a_1, a_2, \ldots. a_k$, and let the remaining letters of I_{e-2}, which may or may not form a single transitive set, be denoted by $b_1, b_2, \ldots.$; we shall speak of the letters a and the letters b.

Letters α, new to I_{e-2}, are connected with letters a by s_{e-1}. Since s_{e-1} is of prime order, any power of s_{e-1} connects a's and α's. If s_{e-1} has two α's in any one cycle, x can be chosen so that in s_{e-1}^x these two new letters are adjacent. Then unless s_{e-1}^x replaces all the k letters a by α's, one of the generators of the group $s_{e-1}^{-x} I_{e-2} s_{e-1}^x$ will connect a's and α's and displace fewer α's than does s_{e-1}, contrary to hypothesis. Suppose that s_{e-1}^x replaces every a by an α, but has two α's in the same cycle with two a's, thus:

$$s_{e-1}^x = (a_1 \alpha_1 \ldots. a_2 \alpha_2 \ldots.) \ldots.$$

We can choose y so that

$$s_{e-1}^{xy} = (a_1 a_2 \ldots\ldots \alpha_1 \alpha_2 \ldots\ldots) \ldots\ldots$$

and proceed as before. It is now evident that if one cycle of s_{e-1} displaces $p-1$ letters α, all the cycles containing a letter a displace $p-1$ α's. In this case

$$s_{e-2} = (a_1 \alpha_1' \alpha_2'' \ldots\ldots \alpha_1^{(p-1)}) (a_2 \alpha_2' \alpha_2'' \ldots\ldots \alpha_2^{(p-1)}) \ldots\ldots (a_k \alpha_k' \ldots\ldots \alpha_k^{(p-1)}) \ldots\ldots,$$

and all the substitutions of the series $s_1, s_2, \ldots\ldots$, not in I_{e-2}, and which connect α's and new letters, are of this form. Since G is a primitive group there is in it some substitution t which replaces an a by an a, and an a by some other letter w. Consider the group $t^{-1} I_{e-2} t$. One of its generators σ connects an a and a letter w. This w cannot be a letter b. Then

$$\sigma = (a' c' c'' \ldots\ldots c^{(p-1)}) \ldots\ldots$$

is of the same type as s_{e-1}, with the k letters a found in k different cycles. This gives

$$t = aa' \ldots\ldots a''c \ldots\ldots ba''' \ldots\ldots$$

Then $tI_{e-2}t^{-1}$ leaves at least one a fixed and has a generator, similar to s_1, which connects letters a with other letters. We can at once conclude that s_{e-1} has not two letters α in any cycle, and further that when $p=2$, s_{e-1} replaces an a by an a. We may write

$$s_e = a_\mu \beta \ldots\ldots,$$

where β is connected with the b's by s_{e-1}, and μ is arbitrary. If s_{e-1} has two letters a, as $a_\mu\ a_{\mu+1}$, adjacent in one of its cycles,

$$s_{e-1}^{-1}\, s_e\, s_{e-1} = a_{\mu+1}\, b' \ldots\ldots$$

Or if s_{e-1} leaves an a fixed, we can make a_μ that letter and have

$$s_{e-1}^{-1}\, s_e\, s_{e-1} = a_\mu\, b \ldots\ldots$$

Hence the theorem as stated is true.

All the primitive groups of class 10 which contain a substitution of degree 10 and order 5 are known.* It remains to determine those which do not contain such a substitution. It is convenient to determine first the diedral groups of class 10 which are generated by two substitutions of the form

$$s_1 = a_1 a_2 \,.\, b_1 b_2 \,.\, c_1 c_2 \,.\, d_1 d_2 \,.\, e_1 e_2 \,.$$

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* Transactions of the American Mathematical Society, Vol. 4 (1903), p. 351.

For brevity D shall be any diedral rotation group. Only one of these groups D is transitive. It is a primitive group of degree 11 and order 22, not contained in a larger primitive group of class 10 of the type we are considering. We neglect it entirely in what follows.

Let s_1 and s_2 be the two generators of a D. The product $s_1 s_2 = S$ is a positive substitution.

If the degree of S is less than the degree of $\{s_1, s_2\}$, s_1 and s_2 have one or more cycles in common. They cannot have three cycles in common for then S would be of degree 8 at most. If s_1 and s_2 have two cycles in common, we have the group

$$D_1 \equiv \{s_1 = a_1 a_2 \,.\, b_1 b_2 \,.\, c_1 c_2 \,.\, d_1 d_2 \,.\, e_1 e_2, \quad s_2 = a_1 a_2 \,.\, b_1 b_2 \,.\, \alpha_1 \alpha_2 \,.\, \beta_1 \beta_2 \,.\, \gamma_1 \gamma_2\}.$$

If s_1 and s_2 have just one cycle in common, S is of degree 12, 13, ... 16. Neglecting the two letters common to s_1 and s_2 we have to do with a positive intransitive D of class 8 and degree 12, 13, ... or 16. S, if of degree 12, is a regular substitution of order 2, 3, or 6. There are two groups:

$$D_2 \equiv \{s_1,\ s_2 = a_1 a_2 \,.\, b_1 c_1 \,.\, b_2 c_2 \,.\, \alpha_1 \alpha_2 \,.\, \beta_1 \beta_2\},$$
$$D_3 \equiv \{s_1,\ s_2 = a_1 a_2 \,.\, b_1 \beta_1 \,.\, c_1 \gamma_1 \,.\, d_1 \delta_1 \,.\, e_1 \varepsilon_1\}.$$

Corresponding to $S^6 = 1$, the D of class 8 cannot be constructed. A substitution of degree 13 contained in G is necessarily regular. If S is of degree 14 and regular, it is of order 7 and there is no group D. If S is not regular, it is of order 4 and $\{s_1, s_2\}$ has three octic constituents. These cannot be so arranged as to give, with the other two constituents, a D generated by two negative substitutions of degree 10. Again no isomorphism can be set up when S is of degree 15. When S is of degree 16 we have

$$D_4 \equiv \{s_1,\ s_2 = a_1 a_2 \,.\, \alpha_1 \alpha_2 \,.\, \beta_1 \beta_2 \,.\, \gamma_1 \gamma_2 \,.\, \delta_1 \delta_2\}.$$

We can now assume that S is of the same degree as $\{s_1, s_2\}$.

If S is of degree 12, it is regular and is of order 2, 3, or 6. We have at once

$$D_5 \equiv \{s_1,\ s_2 = a_1 b_1 \,.\, a_2 b_2 \,.\, c_1 d_1 \,.\, c_2 d_2 \,.\, \alpha_1 \alpha_2 \,.\,\};$$
$$D_6 \equiv \{s_1,\ s_2 = a_1 b_1 \,.\, a_2 c_1 \,.\, b_2 c_2 \,.\, d_1 \alpha_1 \,.\, e_1 \beta_1 \,.\,\},\quad S^3 = 1;$$
$$D_7 \equiv \{s_1,\ s_2 = a_1 b_1 \,.\, b_2 c_1 \,.\, d_1 e_1 \,.\, d_2 \alpha_1 \,.\, e_2 \alpha_2\},\quad S^6 = 1;$$
$$D_8 \equiv \{s_1,\ s_2 = a_1 b_1 \,.\, a_2 \alpha_1 \,.\, b_2 \alpha_2 \,.\, c_1 d_1 \,.\, d_2 e_1\}.$$

The product S cannot be of degree 13. If S is of degree 14, it must be non-regular, and is then of order 4. We have

$$D_9 \equiv \{s_1,\ s_2 = a_1 b_1 \cdot c_1 \alpha_1 \cdot c_2 \alpha_2 \cdot d_1 \alpha_3 \cdot d_2 \alpha_4\},\quad S^4 = 1;$$
$$D_{10} \equiv \{s_1,\ s_2 = a_1 b_1 \cdot c_1 d_1 \cdot e_1 \alpha_1 \cdot e_2 \alpha_2 \cdot \alpha_3 \alpha_4\}.$$

If S is of degree 15:

$$D_{11} \equiv \{s_1,\ s_2 = a_1 \alpha_1 \cdot b_1 \beta_1 \cdot c_1 \gamma_1 \cdot d_1 \delta_1 \cdot e_1 \varepsilon_1\}.$$

If S is of degree 16:

$$D_{12} \equiv \{s_1,\ s_2 = a_1 b_1 \cdot a_2 b_2 \cdot \alpha_1 \alpha_2 \cdot \beta_1 \beta_2 \cdot \gamma_1 \gamma_2\}.$$

The degrees 17, 18, 19 give no groups D.

When S displaces 20 letters, we have,

$$D_{13} \equiv \{s_1,\ s_2 = \alpha_1 \alpha_2 \cdot \beta_1 \beta_2 \cdot \gamma_1 \gamma_2 \cdot \delta_1 \delta_2 \cdot \varepsilon_1 \varepsilon_2\}.$$

Note that D_7 and D_8 contain D_5 and D_6, and that D_9 and D_{10} contain D_2 and D_5.

The D_7, and hence D_8, which is the same group, can be thrown out very quickly. It has two transitive systems of 6 letters each. There is a substitution s_3 of the series $s_1, s_2, \ldots$ which connects these two systems and brings in at most 4 new letters, one in a cycle,* so that $E \equiv \{s_1, s_2, s_3\}$ is a transitive group of degree not greater than 16. We can write s_3 so that it has a cycle new to s_1, is not commutative with it and has more than 4 letters new to s_1. Hence E is of degree 15 at most. If E is of degree 12, it is of order 24 and the only substitutions of degree 10 found in it are the 6 belonging to $\{s_1, s_2\}$. If E is of degree 13, its positive subgroup is of class 12, and contains just one subgroup of order 13, which must then be invariant in the group E. But s_1 cannot transform a group of order 13 into itself. If E is of degree 14, the positive subgroup is of class 12 and of order $14 \cdot 6$. By Sylow's theorem this subgroup has a characteristic cyclic subgroup of order 7, which is then invariant in E. Again s_1 cannot transform a group of order 7 into itself. If E is of degree 15, the positive subgroup is transitive of class 12, and contains no substitutions of degree 13 or 14; hence the subgroup leaving one letter fixed leaves three fixed. Now a group of order $15 \cdot 6$ has a single invariant subgroup of order 5. In E this is impossible. We can now strike D_7 and D_8 from our list.

*Transactions, &c., l. c.

The next to go is the group D_3. The three substitutions of degree 10 in D_3 are

$$s_1 = a_1 a_2 \,.\, b_1 b_2 \,.\, c_1 c_2 \,.\, d_1 d_2 \,.\, e_1 e_2 \,,$$
$$s_2 = a_1 a_2 \,.\, b_1 \beta \,.\, c_1 \gamma \,.\, d_1 \delta \,.\, e_1 \varepsilon \,,$$
$$s_3 = a_1 a_2 \,.\, b_2 \beta \,.\, c_2 \gamma \,.\, d_2 \delta \,.\, e_2 \varepsilon \,.$$

According to the theorem established in the beginning of this article, there is a substitution s, similar to s_1, in the group G, which connects the set $a_1\, a_2$ with another set of D_3. We write $s = a_1 b_1 \ldots.$ If s leaves a_2 fixed we have $s_3 s s_1 s s_3 = a_1 b_1 \,.\, a_2 y \ldots.$ Then putting $s = a_1 b_1 \,.\, a_2 y \ldots,$ we see first that y cannot be new to s_1. For now that D_7 and D_8 are thrown out, a group of the type $\{s_1, s\}$ is not found in the list $D_1, \ldots, D_{13}$. For the same reason y must belong to s_2 and not to s_3. Hence $y = c_1, d_1,$ or e_1. Without loss of generality we can say that $y = c_1$. Now $\{s_1, s\}$ and $\{s_2, s\}$ are both D_6. Hence from $\{s_1, s\}$

$$s = a_1 b_1 \,.\, a_2 c_1 \,.\, b_2 c_2 \ldots.$$

But $(b_2\, c_2)$ is a cycle new to s_2 and on that account $\{s_2, s\}$ cannot be D_6. Hence D_3 can be dropped from our list.

D_6 is next in order of difficulty. The three substitutions of order 2 are

$$s_1 = a_1 a_2 \,.\, b_1 b_2 \,.\, c_1 c_2 \,.\, d_1 d_2 \,.\, e_1 e_2 \,,$$
$$s_2 = a_1 b_1 \,.\, a_2 c_1 \,.\, b_2 c_2 \,.\, d_1 \alpha_1 \,.\, e_1 \alpha_2 \,,$$
$$s_3 = a_2 b_2 \,.\, a_1 c_2 \,.\, b_1 c_1 \,.\, d_2 \alpha_1 \,.\, e_2 \alpha_2 \,.$$

There is a substitution s similar to s_1, which connects the set $a_1 \ldots$ with the set $d_1 d_2 \alpha_1$, and brings in at most 4 new letters, one to a cycle. Suppose first that $E \equiv \{D_6, s\}$ is transitive. If E is of degree 12, it has 4 subgroups of order 3 and is an imprimitive group isomorphic to the symmetric group of degree 4. In $(abcd)$ *all* we get a transitive representation of class 10 on 12 letters only by taking as a new element in the multiplication table of the group a subgroup 1, (ab). This substitution (ab) is contained in one subgroup of order 4, in two of order 6, in one of order 8, and in one of order 12. Hence by MARGGRAFF'S extension of a theorem* due to JORDAN, this group E, if contained in a primitive group, is contained in a doubly transitive group of degree 13, of order $13 \cdot 12 \cdot 2$.

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*MARGGRAFF, *Dissertation, Ueber primitive Gruppen mit transitiven Untergruppen geringeren Grades.* Giessen (1889). This point is made clear in a paper soon to appear in the Transactions of the American Mathematical Society under the title *On Multiply Transitive Groups*, by the present writer.

This G^{13} would by Sylow's theorem have to have an invariant subgroup of order 13. This is here impossible. It is clear that E cannot be of degree 13, 14, 15, or 16, if we keep in mind the reasoning in the case of D_7.

Now s does not connect more than two systems of D_6. By means of the conjoin of the regular constituent of D_6 we can write without loss of generality

$$s = a_1 \alpha_1 . \dots .$$

Looking over the non-Abelian groups in the list $D_1, \dots$, we see that s can have 2, 4, or 5 letters new to s_1.

If s has just two letters new to s_1, $\{s_1, s\}$ is D_6 and

$$s = a_1 \alpha_1 . e_i x \dots (a_2)(e_j),$$

with all the 6 letters b, c, d displaced. We now arrive at a contradiction. If $e_i = e_1$, by $\{s_2, s\}$, $x \neq \varepsilon$, and by $\{s_3, s\}$, $x = \varepsilon$; there is a similar contradiction if $e_i = e_2$.

Next let s have just 4 letters new to s_1. Here

$$s = a_1 \alpha_1 . a_2 x_1 . {\textstyle\frac{b}{c}} x_2 . {\textstyle\frac{b}{c}} x_3 . {\textstyle\frac{cd}{bd}},$$

or

$$s = a_1 \alpha_1 . a_2 x_1 . e_1 x_2 . e_2 x_3 . cd.$$

Comparison with s_2 and s_3 shows that these forms for s have to be rejected because of the number of new letters.

If s has 5 letters new to s_1,

$$s = a_1 \alpha_1 . b x_1 . c x_2 . d x_3 . e x_4.$$

Clearly $x_4 \neq \alpha_2$. Then neither $\{s_2, s\}$ nor $\{s_3, s\}$ is possible. This clears D_6 out of the way.

D_9, and with it D_{10}, goes out quickly. There are 4 substitutions in D_9 of order 2:

$$s_1 = a_1 a_2 . b_1 b_2 . c_1 c_2 . d_1 d_2 . e_1 e_2, \qquad s_2 = a_1 b_1 . c_1 \alpha_1 . c_2 \alpha_2 . d_1 \alpha_3 . d_2 \alpha_4$$
$$s_3 = a_1 b_2 . a_2 b_1 . \alpha_1 \alpha_2 . \alpha_3 \alpha_4 . e_1 e_4, \qquad s_4 = a_2 b_2 . c_2 \alpha_1 . c_1 \alpha_2 . d_2 \alpha_3 . d_1 \alpha_4.$$

The substitution s may have two forms:

$$(1)\ s = e_1 a_1 \dots, \qquad (2)\ s = e_1 c_1 \dots.$$

Consider $s = e_1 a_1 \dots$. The groups $\{s_1, s\}$ and $\{s_3, s\}$ are D_9 or D_{10}. In either case

$$s = a_1 e_1 \dots (a_2)(e_2)(b_2).$$

If $\{s_1, s\}$ is D_9, $\{s_3, s\}$ is also D_9, and $s = a_1 e_1 . c_1\alpha' . c_2\alpha'' . d_1\alpha''' . d_2\alpha^{\mathrm{IV}}$, with only one letter, e_1, new to s_2; $\{s, s_2\}$ is not in our list. If $\{s_1, s\}$ is D_{10}

$$s = a_1 e_1 . b_1 x \ldots ,$$

where x is a c or d. But $\{s_3, s\}$ is also D_{10}, and x is an α. Consider the second case, $s = e_1 c_1 \ldots$ If $ss_1 = s_1 s$, we have uniquely:

$$s = c_1 e_1 . c_2 e_2 . a_1 a_2 . \alpha_3 \beta_1 . \alpha_4 \beta_2 .$$

The group $\{s_1, s_2, s\}$ has three sets of intransitivity. We now have, going one step further, either

$$(1)\ s' = e_1 d_1 . e_2 d_2 . b_1 b_2 \ldots ,$$

an impossibility on account of $\{s, s'\}$, or

$$(2)\ s' = e_1 \beta_1 \ldots$$

Now s' must either add new letters to the set $a_1 \ldots$ or connect $d_1 \ldots$ and $a_1 \ldots$ In neither case can $s' = e_1 \beta_1 \ldots$ be constructed. Hence D_9 and D_{10} can be dropped from our list.

Only one non-Abelian D is left. It is D_{11}. If a substitution s of the series $s_1, \ldots$ connects two cycles of another substitution s' of the series, s and s' are commutative.

We now take up D_2. It is generated by

$$s_1 = a_1 a_2 . b_1 b_2 . c_1 c_2 . d_1 d_2 . e_1 e_2 ,$$

and

$$s_2 = a_1 a_2 . b_1 c_1 . b_2 c_2 . \alpha_1 \alpha_2 . \beta_1 \beta_2 .$$

There are two substitutions, (1) $s = a_1 b_1 \ldots$, (2) $s = a_1 d_1 \ldots$ to be considered in extending D_2. Let $s = a_1 b_1 \ldots$ Neither $ss_1 = s_1 s$ nor $ss_2 = s_2 s$ is possible. If $s = a_1 d_1 \ldots$, then $ss_1 = s_1 s$. But $\{s_2, a_1 d_1 . a_2 d_2 \ldots\}$ is impossible. Hence G never includes D_2.

The generators of D_5 are s_1 and

$$s_2 = a_1 b_1 . a_2 b_2 . c_1 d_1 . c_2 d_2 . \alpha_1 \alpha_2 .$$

There are again only two substitutions s to be examined. The first,

$$s = a_1 \alpha_1 . b_1 \alpha_2 . c_1 \beta_1 \ldots ,$$

we reject at once. But the second,

$$s = a_1 c_1 . a_2 c_2 . b_1 d_1 . b_2 d_2 . \beta_1 \beta_2 ,$$

requires that the next group $\{s_1, s_2, s, s'\}$ be considered. Now s' must be $a_1 \alpha_1 \ldots$, an impossibility. D_5 is rejected. We throw out D_1 also. In it

$$s_2 = a_1 a_2 \,.\, b_1 b_2 \,.\, \alpha_1 \alpha_2 \,.\, \beta_1 \beta_2 \,.\, \gamma_1 \gamma_2 \,.$$

Obviously we cannot have $s = a_1 c_1 \,.\, a_2 c_2 \ldots$. If

$$s = a_1 b_1 \,.\, a_2 b_2 \,.\, \delta_1 \delta_2 \,.\, \varepsilon_1 \varepsilon_2 \,.\, \zeta_1 \zeta_2 \,,$$

we pass step by step by means of the substitutions

$$\begin{aligned} s' &= a_1 \delta_1 \,.\, b_1 \delta_2 \,.\, c_1 \alpha_1 \,.\, d_1 \beta_1 \,.\, e_1 \gamma_1 \,, \\ s'' &= a_1 \varepsilon_1 \,.\, b_1 \varepsilon_2 \,.\, c\alpha' \,.\, d\beta' \,.\, e\gamma' , \\ s''' &= a_1 \zeta_1 \,.\, b_1 \zeta_2 \,.\, c\alpha' \,.\, d\beta' \,.\, e\gamma' , \; (\alpha' = \alpha_1 \text{ or } \alpha_2, \text{ \&c.}) \end{aligned}$$

to an intransitive group $\{s_1, \ldots s'''\}$, with which we must stop. Then D_1 may be rejected.

It is now clear that D_4 also may be struck from our list.

Only D_{11}, D_{12}, and D_{13} are left.

Consider D_{11}. It leads to two primitive groups of class 10. The substitutions of order 2 in D_{11} are s_1,

$$\begin{aligned} s_2 &= a_1 a_3 \,.\, b_1 b_3 \,.\, c_1 c_3 \,.\, d_1 d_3 \,.\, e_1 e_3 \,, \\ s_3 &= a_2 a_3 \,.\, b_2 b_3 \,.\, c_2 c_3 \,.\, d_2 d_3 \,.\, e_2 e_3 \,. \end{aligned}$$

There must be in $s_1, \ldots$ a fourth substitution s_4 non-commutative with two of these three, non-commutative with s_1 and s_2, say. This follows from the fact that $\{s_1, \ldots\}$ is transitive. There are only two forms we need assume for s_4:

$$s_4 = a_1 a_4 \,.\, b_1 b_4 \,.\, c_1 c_4 \,.\, d_1 d_4 \,.\, e_1 e_4 \,, \tag{1}$$

$$s_4 = a_2 b_3 \,.\, a_3 b_2 \,.\, c_1 c_4 \,.\, d_1 d_4 \,.\, e_1 e_4 \,. \tag{2}$$

Consider (2). The group $E \equiv \{D_{11}, s_4\}$ has in all 6 substitutions of degree 10. The two not yet written down are:

$$\begin{aligned} s_5 &= a_1 b_3 \,.\, a_3 b_1 \,.\, c_2 c_4 \,.\, d_2 d_4 \,.\, e_2 e_4 \,, \\ s_6 &= a_2 b_1 \,.\, a_1 b_2 \,.\, c_3 c_4 \,.\, d_3 d_4 \,.\, e_3 e_4 \,. \end{aligned}$$

There is in E a set of intransitivity of 6 letters, $a_1 \ldots$, so that we may extend E by means of

$$s_7 = a_1 c_1 \,.\, a_2 c_2 \,.\, a_3 c_3 \,.\, d_4 d_5 \,.\, e_4 e_5 \,,$$

or

$$a_1 c_3 \,.\, a_3 c_1 \,.\, b_2 c_4 \,.\, d_2 d_5 \,.\, e_2 e_5 \,.$$

These two substitutions are conjugate under

$$t = a_2 b_2 \,.\, c_1 c_3 \,.\, c_2 c_4 \,.\, d_1 d_3 \,.\, d_2 d_4 \,.\, e_1 e_3 \,.\, e_2 e_4 \,,$$

which also transforms s_2 into s_2, s_1 into s_6, and s_3 into s_4. Hence only the first

form of s_7 need be examined. Beside the substitutions already written down $E' = \{E, s_7\}$ has the following three substitutions similar to s_1:

$$s_8 = a_1 c_4 \,.\, b_3 c_2 \,.\, b_2 c_3 \,.\, d_1 d_5 \,.\, e_1 e_5,$$
$$s_9 = b_1 c_1 \,.\, a_2 c_4 \,.\, b_1 c_3 \,.\, d_2 d_5 \,.\, e_2 e_5,$$
$$s_{10} = b_2 c_1 \,.\, b_1 c_2 \,.\, a_3 c_4 \,.\, d_3 d_5 \,.\, e_3 e_5.$$

In extending E', two substitutions s_{11} are to be considered. The one, $s_{11} = a_1 d_3 \ldots$ is impossible. The other

$$s_{11} = a_1 d_1 \,.\, a_2 d_2 \,.\, a_3 d_3 \,.\, c_4 d_5 \,.\, e_4 e_6,$$

gives the group E'', in which the other substitutions of order 2 and class 10 are

$$s_{12} = a_1 d_4 \,.\, b_3 d_2 \,.\, b_2 d_3 \,.\, c_1 d_5 \,.\, e_1 e_6,$$
$$s_{13} = a_2 d_4 \,.\, b_3 d_1 \,.\, b_1 d_3 \,.\, c_2 d_5 \,.\, e_2 e_6,$$
$$s_{14} = a_3 d_4 \,.\, b_1 d_2 \,.\, b_2 d_1 \,.\, c_3 d_5 \,.\, e_3 e_6,$$
$$s_{15} = c_1 d_1 \,.\, c_2 d_2 \,.\, c_3 d_3 \,.\, c_4 d_4 \,.\, e_5 e_6.$$

Finally

$$s_{16} = a_1 e_1 \,.\, a_2 e_2 \,.\, a_3 e_3 \,.\, c_4 e_5 \,.\, d_4 e_6$$

is unique. The transitive group $\{s_1, s_2, s_4, s_7, s_{11}, s_{16}\}$ is primitive of degree 21, and is isomorphic to the symmetric group on 7 letters. The subgroup leaving one letter fixed is $\{E', s_{16} s_{11} s_{16} = d_1 e_1 \,.\, d_2 e_2 \,.\, d_3 e_3 \,.\, d_4 e_4 \,.\, d_5 e_5\}$. This subgroup has two sets of intransitivity. It is of order $2\,(5\,!)$. Hence the isomorphism between $\{E'', s_{16}\} \equiv E'''$ and $(abcdefg)$ *all* is simple.

This group E''' cannot be contained in a doubly transitive group, for then we should have a substitution

$$s = a_1 e_6 \,.\, e_1 d_4 \,.\, d_1 e_4 \,.\, b_1 \text{—} \,.\, c_1 \text{—},$$

similar to s_1, which cannot exist at the same time as s_4. This also means that no group containing E''' can have other substitutions similar to s_1. Nor can E''' be invariant in a larger group of degree 21.

There remains the first form of s_4:

$$s_4 = a_1 a_4 \,.\, b_1 b_4 \,.\, c_1 c_4 \,.\, d_1 d_4 \,.\, e_1 e_4:$$

Only one form for s_7 is possible:

$$s_7 = a_1 a_5 \,.\, b_1 b_5 \,.\, c_1 c_5 \,.\, d_1 d_5 \,.\, e_1 e_5.$$

Continuing as before we get the unique set of generating operators, s_1, s_2, s_4, s_7, and

$$s_1' = a_1 b_1 \,.\, a_2 b_2 \,.\, a_3 b_3 \,.\, a_4 b_4 \,.\, a_5 b_5 ,$$
$$s_2' = a_1 c_1 \,.\, a_2 c_2 \,.\, a_3 c_3 \,.\, a_4 c_4 \,.\, a_5 c_5 ,$$
$$s_3' = a_1 d_1 \,.\, a_2 d_2 \,.\, a_3 d_3 \,.\, a_4 d_4 \,.\, a_5 d_5 ,$$
$$s_4' = a_1 e_1 \,.\, a_2 e_2 \,.\, a_3 e_3 \,.\, a_4 e_4 \,.\, a_5 e_5 .$$

These give an imprimitive group H of order $(5\,!)^2$, and of degree 5^2. It is not possible to form another substitution s similar to s_1. Hence H is invariant in any primitive group of class 10 containing it. Adjoining the substitution

$$t = a_2 b_1 \,.\, a_3 c_1 \,.\, a_4 d_1 \,.\, a_5 e_1 \,.\, b_3 c_2 \,.\, b_4 d_2 \,.\, b_5 e_2 \,.\, c_4 d_3 \,.\, c_5 d_3 \,.\, d_5 e_4 ,$$

we have a primitive group $G \equiv \{H, t\}$ of class 10, degree 25, and order $2\,(5\,!)^2$.

For all integral values of k greater than 2 there exists a primitive group G of degree k^2 and class $2\,k$ with an invariant subgroup H of the same type as H^{25}. Thus the imprimitive group H is generated by

$$\begin{aligned}
s_2 &= (a_1 a_2)\,(b_1 b_2)\,(c_1 c_2) \cdots\cdots (j_1 j_2)\,(k_1 k_2),\\
s_3 &= (a_1 a_3)\,(b_1 b_3)\,(c_1 c_3) \cdots\cdots (j_1 j_3)\,(k_1 k_3),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
s_k &= (a_1 a_k)\,(b_1 b_k)\,(c_1 c_k) \cdots\cdots (j_1 j_k)\,(k_1 k_k),\\
\sigma_b &= (a_1 b_1)\,(a_2 b_2)\,(a_3 b_3) \cdots\cdots (a_j b_j)\,(a_k b_k),\\
\sigma_c &= (a_1 c_1)\,(a_2 c_2)\,(a_3 c_3) \cdots\cdots (a_j c_j)\,(a_k c_k),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\sigma_k &= (a_1 k_1)\,(a_2 k_2)\,(a_3 k_3) \cdots\cdots (a_j k_j)\,(a_k k_k).
\end{aligned}$$

If now we adjoin to H the substitution

$$\begin{aligned}
t = (a_1)\,(a_2 b_1)\,(a_3 c_1)\,(a_4 d_1) \cdots (a_j j_1)\,(a_k k_1)&\\
(b_2)\,(b_3 c_2)\,(b_4 d_2) \cdots (b_j j_2)\,(b_k k_2)&\\
(c_3)\,(c_4 d_3) \cdots (c_j j_3)\,(c_k k_3)&\\
\cdots\cdots\cdots\cdots&\\
(j_j)\,(j_k k_j)&\\
(k_k),&
\end{aligned}$$

the group $G \equiv \{H, t\}$ is primitive since the subgroup

$$\{s_2, s_3, \ldots\, s_j, \sigma_b, \sigma_c, \ldots\, \sigma_j, t\},$$

of order $2[(k-1)!]^2$ and omitting k_k, is maximal when k is greater than 2.

This group G is the only transitive group in which H is invariant.

If a second transitive group (G') exists in which H is invariant, there is a primitive group $G'' \equiv \{G, G'\}$ of order greater than $2(k!)^2$ in which H is invariant. Let G_1, H_1, ... be the subgroups of G, H, ... respectively, which leave one letter fixed. Since systems of imprimitivity of H can be chosen in only two ways, G'' is simply transitive.* Then G_1'' is intransitive, and since it includes G_1, the two sets of intransitivity of the latter are also the sets of intransitivity of G_1''. Now the larger set of intransitivity of G_1 is exactly that group of the system we are considering which corresponds to the value $k-1$ of k. Hence if the theorem is true for one value of k, it is true for the next succeeding value. But it certainly holds for $k=3$. Therefore the theorem is true in general.

The transitive group generated by the complete set of conjugates s_1, ... cannot be Abelian. Then our task is completed. There are three primitive groups of class 10 which do not contain a substitution of order 5 on 10 letters: the diedral G^{11}_{22}; the $G^{21}_{7!}$ isomorphic to the symmetric group on 7 letters; and the new group $G^{25}_{2(5!)^2}$. We recall the four groups found in a previous paper:† the metacyclic $\pm\, G^{11}_{11\cdot 10}$, the Mathieu $\pm\, G^{12}_{12\cdot 11\cdot 10}$, and their positive subgroups. Two of these groups, the G^{21} and the G^{25}, are omitted by JORDAN‡ and one group is in his list for class 10 which does not belong there. All of these groups belong to infinite systems. The system to which G^{21} belongs is that of the symmetric-k group written on $\frac{k(k-1)}{2}$ letters. These groups are of class $2k-4$ (not $2k-2$ as JORDAN has it) and are primitive if k is greater than 4.

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* *On the Primitive Groups of Class Eight.* Bulletin of the American Mathematical Society (1904), 2nd Ser., v. 10, p. 286. An invariant imprimitive subgroup of a multiply transitive group is at least three-fold imprimitive.

† Transactions of the American Mathematical Society, l. c.

‡ Comptes Rendus. Vol. 75 (1872), p. 1754.